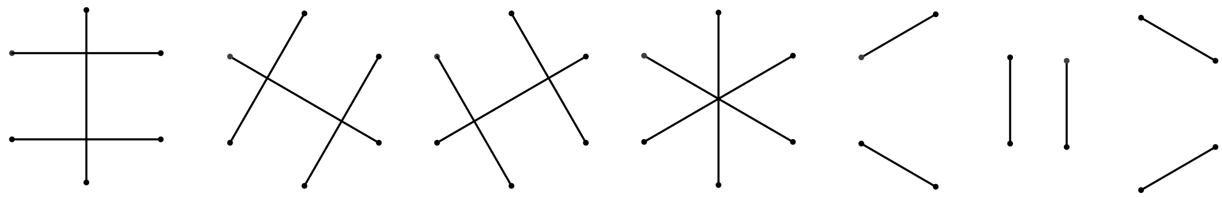
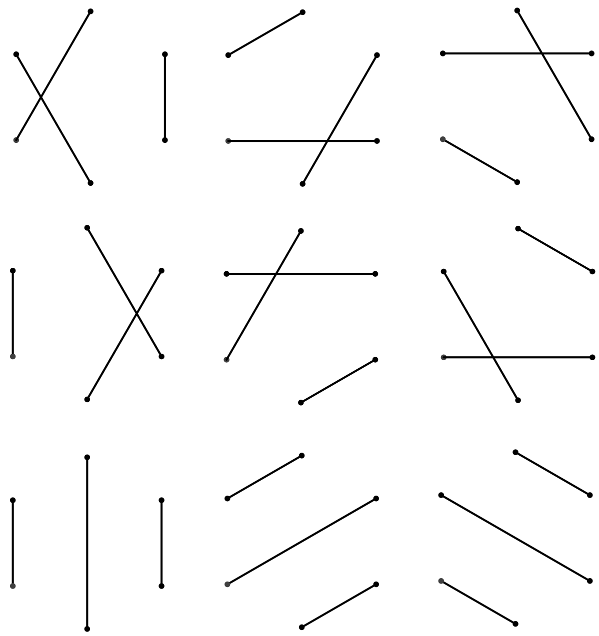
Synthematics: Solution

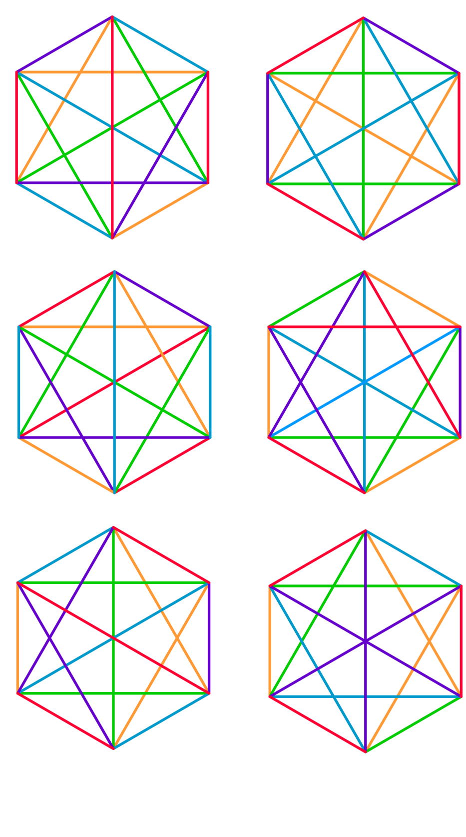




Three line segments sharing no vertex can be constructed one segment at a time: $\binom{6}{2} = 15$ options for the first segment, $\binom{4}{2} = 6$ options for the second, then the third is determined by the first two. But three segments can be chosen in any of $3! = 6$ possible orders, so there are $\binom{6}{2}\binom{4}{2}/3! = 15$ triples of segments sharing no vertex.

Imagine constructing the top right synthematic to the right. There are 15 options for the red triple, then 8 disjoint from the red to choose the purple. Only 4 options left, the top left triples above, but choosing the snowflake prevents us from choosing any more, so the last three triples are determined. The shape is irrelevant to the counting process - by symmetry, the number of options for each choice does not depend on previous choices. The triples can be chosen 5! orders, so the answer is

$$\frac{15 \cdot 8 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 6.$$



The vertices may be labelled one through six. The line segments, or **duads**, then, are subsets $\{a, b\}$. The triples are partitions of $\{1, 2, 3, 4, 5, 6\}$ of the shape $\{\{a, b\}, \{c, d\}, \{e, f\}\}$, called **synthemes** by Sylvester. The number of synthemes can be expressed in terms of multinomial coefficients as $\frac{1}{3!}\binom{6}{2,2,2}$. The colorful hexagons, called **synthematic totals**, are partitions of set of all 15 line segments $E = \{\{a, b\} \mid 1 \le a < b \le 6\}$ into 5 synthemes.

A permutation of the six vertices induces a permutation of the six synthematic totals. Swapping two vertices winds up swapping three pairs of totals (and vice-versa), and cycling three vertices winds up cycling two sets of three totals (and vice-versa). In the language of group theory, this exhibits the unique nontrivial outer automorphism of the symmetric group .

The hexagon with all its line segments forms a complete graph K_6. This can be considered a **hemi-icosehedron** (a kind of projective polyhedron), since an icosahedron has 6 opposite pairs of vertices and 15 opposite pairs of edges. Synthemes correspond to inscribed compounds of orthogonal golden rectangles. The symmetry group of the octahedron, A_5, is an **exotic copy** of the usual alternating subgroup $A_5 \subset S_6$.

The fact $\operatorname{Out} S_6 \cong \mathbb{Z}_2$ has order two indicates there is *duality*. Indeed, constructing the set of synthematic totals out of a six element set can be considered (the restriction of a) **combinatorial species** which has order two under composition. Moreover, the duads and synthemes are the vertices and edges of the self-dual **Cremona-Richmond configuration**.